Turbo-like Codes over the Block Fading Channel: Code Design and Construction

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Abstract: *Transmission of turbo-like codes over block fading channels is addressed. We present a novel construction based on blockwise concatenation that yields maximum distance separable turbo-like codes. We emphasize the importance of blockwise concatenation with respect to standard (serial, parallel or hybrid) constructions that perform well on the fully interleaved (ergodic) fading channel, but may suffer severe degradation in the block fading (non-ergodic) case. We analyze the performance of the proposed codes by means of upper bounds on the maximum-likelihood (ML) decoding error probability and we show that iterative decoding performs very close to ML even for short block lengths. We discuss how the proposed codes approach outage probability and we show that blockwise concatenated codes perform close to outage probability also in the case of long block lengths.*

Keywords: Turbo codes, block fading channels.

1. INTRODUCTION

Coding for block-fading channels has recently become an important research area mainly due to the wide range of applications to real wireless communication systems (see [1] and references therein). Among others, orthogonal frequency division multiplexing (OFDM), frequency-hopped systems, multiple antenna transmission schemes with suboptimal detection, can be modeled as instances of a block-fading channel. On ergodic fading channels, where there is infinite diversity, capacity approaching codes, such as turbo-like codes or low-density parity-check (LDPC) codes have been shown to be very effective (e.g. [2]). However codes designed for ergodic channels need not be effective over block fading channels.

Much of previous work on code construction for block fading channels [1] has mainly concentrated on convolutional codes (CC), due to their simple construction, low decoding complexity and their ability to perform close to the information outage probability for relatively short block length. However, turbo-like codes [3] have not yet deserved much attention because there was no general construction allowing to achieve the diversity/rate trade-off offered by the Singleton Bound [1]. Only simple configurations of rate-constrained parallel concatenated convolutional codes (PCCC) have been considered [4].

In this paper, we present a simple general construction of turbo-like codes that can achieve the Singleton bound. We nickname this family of codes as blockwise concatenated codes (BCC), and we study some of their properties. In particular, we illustrate the relevance of the blockwise concatenation, and we show the advantage of BCCs over standard turbo-like codes designed for the ergodic fading channel. We also analyze their performance with maximum-likelihood (ML) decoding error probability upper bounds, giving an explicit way to compute the weight enumerators and we provide simulation results with iterative decoding. We show that in a block fading environment, ML decoding and iterative decoding perform very close to each other even for short block lengths, and that there is no advantage in using improved bounding techniques as for ergodic channels [5], [2]. BCCs are also shown to approach outage probability in a *fair* way, i.e., they show a frame error rate (FER) performance basically independent of the block length. Moreover, they are shown to provide substantial gain over convolutional codes for medium to large block length.

2. SYSTEM MODEL

We consider asingle-input single-output block-fading channel with N_B fading blocks, such that the received signal at block b $\mathbf{y}_b \in \mathbb{C}^{L_B}$ is given by,

$$\mathbf{y}_b = \sqrt{E_s} h_b \mathbf{x}_b + \mathbf{z}_b \ , \ b = 1, \dots, N_B \tag{1}$$

where $\mathbf{x}_b \in \{-1,+1\}^{L_B}$ is the BPSK transmitted signal at block b with L_B the codeword length per block, E_s is the symbol energy, h_b is the b-th fading channel coefficient with $\mathbf{h} = (h_1, \dots, h_{N_B})$, and $\mathbf{z}_b \in \mathbb{C}^{L_B}$ is the vector of complex noise samples i.i.d. $\sim \mathcal{N}(0, N_0)$. We denote the codeword length by $L = N_B L_B$. The channel coefficients are assumed to be i.i.d. circularly symmetric Gaussian random variables (Rayleigh fading). We assume that the channel states are perfectly known to the receiver and not known at the transmitter.

This model can represent general time/frequency selective fading channels where the blocks are identified with time/frequency slots. Fading blocks may be either independent or correlated. For the case of independent blocks the two extreme cases of quasistatic and fully interleaved channels can be obtained from (1) by letting $N_B = 1$ and $N_B = L$ respectively. For the case of corre-

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lated fading blocks , the model (1) is well suited to represent a slowly-varying frequency-selective channel, where fading blocks are identified with the sub-carriers of an OFDM system. Eq. (1) can also represent a multiple antenna system where the intereference among the different antennas has been perfectly removed. In this case, the performance of such a system can be used as benchmark corresponding to the space-time matched-filter bound, for sub-optimal detection strategies based on interference cancellation [6]. Frequency-hopped systems can also be represented by (1).

3. CODE DESIGN

The average Pairwise Error Probability (PEP), i.e., the average (over the channel states) probability of decoding in favor of $\mathbf{x}'$, when $\mathbf{x}$ has been transmitted, is given by,

$$\mathrm{E}[P(\mathbf{x} \to \mathbf{x}'|\mathbf{h})] \leq \frac{1}{2} \prod_{b=1}^{N_B} \frac{1}{1 + \frac{E_s}{N_0} w_b}. \qquad (2)$$

w_b is the Hamming weight of block b. We define the *block diversity* of a binary code $\mathcal{C}$ mapped onto N_B blocks as the blockwise Hamming distance, $\delta_\beta = \min_{\mathbf{x} \in \mathcal{C}} |\{b \in [1, \ldots, N_B] \, : \, w_b \neq 0\}|$. Then, as apparent from (2), the diversity performance at high SNR is dominated by δ_β and we will be interested in maximizing δ_β, i.e. the number of nonzero rows of $\mathbf{X} - \mathbf{X}'$, with $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_{N_B}]^T$. As known from [1], the achievable diversity when mapping a binary code of rate r over N_B fading blocks is given by the Singleton bound (SB) on the block diversity, $\delta_\beta \leq 1 + \lfloor N_B (1 - r) \rfloor$, and the codes achieving the bound with equality are named maximum distance separable (MDS) codes. Consequently, in our design we will search for MDS codes, i.e., those maximizing δ_β achieving the SB for all values of N_B.

4. CODE CONSTRUCTION

In this paper, we propose a general structure based on a blockwise concatenation of an binary outer code $\mathcal{C}^O \in \mathbb{F}_2^{L_O}$ of length L_O and rate r_O cyclically mapped over N_B blocks, with N_B binary inner encoders $\mathcal{C}^I \in \mathbb{F}_2^{L_I}$ of length L^I and rate r_I, through N_B interleaver permutations $\boldsymbol{\pi}_b, \; b = 1, \ldots, N_B$ of length $L_\pi = L_O / N_B$, as illustrated in Figure 1.

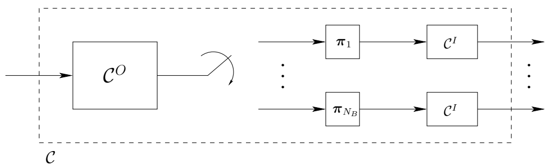

Figure 1: Code structure for BCCs.

The rate of the resulting BCC $\mathcal{C}^{BCC} \in \mathbb{F}_2^L$ is given by $r_{BCC} = r_O r_I$, where $L = L_I N_B$. When the outer code is a simple repetition code of rate $r_O = 1/N_B$ and the inner codes are rate one accumulators, we will refer to the resulting structure as repeat and *blockwise* accumulate (RBA) codes. When both the outer and inner codes are convolutional codes, we will refer to the resulting structure as *blockwise* concatenated convolutional codes (BCCC) or blockwise turbo-codes. Since interleavers and inner encoding are performed on a blockwise basis, the diversity properties of the outer code are respected and $\delta_\beta(\mathcal{C}) = \delta_\beta(\mathcal{C}^O)$. Therefore, in order to optimize the performance of BCCs, we will search for MDS outer codes.

5. DECODING

By applying the sum-product algorithm to the BCC dependency graph, the decoder reduces to N_B maximum-a-posteriori (MAP) soft-input soft-output (SISO) decoders of the inner codes $\mathcal{C}^I$ and a MAP SISO decoder of $\mathcal{C}^O$ that exchange extrinsic information messages $\mu_{I_b \to O}$ (from the b-th $\mathcal{C}^I$ decoder to $\mathcal{C}^O$ decoder) and $\mu_{O \to I_b}$ (from the $\mathcal{C}^O$ decoder to b-th $\mathcal{C}^I$ decoder) over the iterations, in a similar way to what it is done to decode parallel concatenated codes (PCC) and serial concatenated codes (SCC).

6. ML PERFORMANCE ANALYSIS

In the case of the block-fading channel, analysis of iterative decoding may be a very difficult task, due the random multivariate nature of the problem, i.e., for every set of channel states $h_1, \ldots, h_{N_B}$, the convergence properties of the iterative decoder will be different. We therefore resort to maximum-likelihood (ML) error probability analysis.

6.1. ML Upper Bounds

A tight upper bound to the frame error probability P_F of binary codes mapped over N_B blocks based on the union bound and an *limit before average* approach, is given by [7],

$$P_F \leq \mathrm{E}\left[\min\left\{1, \sum_{w_1, \ldots, w_{N_B}} A_{w_1, \ldots, w_{N_B}} \mathrm{Q}\left(\sqrt{\frac{E_s d^2}{2N_0}}\right)\right\}\right] \qquad (3)$$

where $d^2 = 4E_s \sum_{b=0}^{N_B} |h_b|^2 w_b$ is the squared Euclidean distance of a pairwise error event with weights $w_1, \ldots, w_{N_B}$, $A_{w_1, \ldots, w_{N_B}}$ is the multivariate weight enumeration function (MWEF) of $\mathcal{C}$ which accounts for the number of pairwise error events with output Hamming weights per block $w_1, \ldots, w_{N_B}$, and $\mathrm{Q}(x) = 1/\sqrt{2\pi} \int_x^\infty e^{-(t^2/2)} dt$ is the Gaussian tail function. E[.] denotes average with respect to the fading statistics.

Union bound-based techniques are known not to provide good estimates of the error probability of turbo-codes over AWGN or ergodic fading channels. On the contrary, the tangential-sphere bound [8], [5] was shown to provide very accurate results. In our case, the tangential-sphere bound on P_F is given by,

$$\begin{aligned} P_F \leq \mathrm{E}\Bigg[\int_{-\infty}^{+\infty} & \frac{dz_1}{\sqrt{2\pi\sigma^2}} e^{-z_1^2/2\sigma^2} \Bigg\{1 - \bar{\Gamma}\left(\frac{L-1}{2}, \frac{r_{z_1}}{2\sigma^2}\right) + \\ & + \sum_{\substack{w_1, \ldots, w_{N_B} \\ d:\, d/2 < \alpha}} A_{w_1, \ldots, w_{N_B}} \bar{\Gamma}\left(\frac{L-2}{2}, \frac{r_{z_1}^2 - \beta_d(z_1)^2}{2\sigma^2}\right) \cdot \\ & \cdot \left[\mathrm{Q}\left(\frac{\beta_d(z_1)}{\sigma}\right) - \mathrm{Q}\left(\frac{r_{z_1}}{\sigma}\right)\right]\Bigg\}\Bigg], \end{aligned} \qquad (4)$$

where $\bar{\Gamma}(a, x) = \frac{1}{\Gamma(a)} \int_0^x t^{a-1} e^{-t} dt$ is the normalized incomplete gamma function and $\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt$ is the gamma function, $\sigma^2 = N_0/2$, $r_{z_1} = r\big(1 - z_1/R\big)$, $\beta_d(z_1) = \frac{r_{z_1}}{\sqrt{1 - d^2/R^2}} \frac{d}{2r}$, $\alpha_d = r\sqrt{1 - d^2/R^2}$, $R^2 = E_s L_B \sum_{b=1}^{N_B} |h_b|^2$ and r, the cone radius, is the solution of $\sum_{\substack{w_1,\ldots,w_{N_B} \\ d:\, d/2<\alpha}} A_{w_1,\ldots,w_{N_B}} \int_0^{\theta_k} \sin^{N-3} \phi d\phi = \frac{\sqrt{\pi}\Gamma\left(\frac{L-2}{2}\right)}{\Gamma\left(\frac{L-1}{2}\right)}$ with $\theta_k = \cos^{-1}\left(\frac{d}{2r} \frac{1}{\sqrt{1 - d^2/R^2}}\right)$. Notice that no hard limiting to 1 before average is needed here, since the argument of the expectation in (4) is a probability.

6.2. Weight Enumerators

Now, in order to compute (3) and (4), we need to find the MWEFs for the BCC structure presented above. To compute the multivariate weight enumerators, we assume that blockwise concatenation is performed through a set of N_B uniform interleavers, and we compute the average multivariate weight enumeration function as,

Proposition 1 *Let $\mathcal{C}^{BCC}$ be a blockwise concatenated code mapped over N_B fading blocks constructed by concatenating an outer code $\mathcal{C}^O$ mapped over N_B blocks with input multivariate-output weight enumeration function $A^O_{i,w_1,\ldots,w_{N_B}}$, and N_B inner codes $\mathcal{C}^I$ with input-output weight enumeration functions $A^I_{i,w}$, through N_B uniform interleavers of length L_π. Then, the average input multivariate-output weight enumeration function of $\mathcal{C}^{BCC}$, $A^{BCC}_{i,w_1,\ldots,w_{N_B}}$, is given by,*

$$A^{BCC}_{i,w_1,\ldots,w_{N_B}} = \sum_{\ell_1,\ldots,\ell_{N_B}} \frac{A^O_{i,\ell_1,\ldots,\ell_{N_B}} \prod_{b=1}^{N_B} A^I_{\ell_b,w_b}}{\prod_{b=1}^{N_B} \binom{L_\pi}{\ell_b}}. \quad (5)$$

Notice that in order to compute upper bounds on the frame error probability P_F we need $A_{w_1,\ldots,w_{N_B}} = \sum_i A_{i,w_1,\ldots,w_{N_B}}$.

7. NUMERICAL EXAMPLES

In this section we report some numerical examples of the frame error rate performance of BCCs. In particular, we consider BCCCs and RBAs and we discuss some of their interesting properties. Simulations use a different interleaver every frame (average interleaver).

7.1. Importance of the Blockwise Concatenation

Figure 2 compares the FER performance computed by simulation of $r = 1/2$ RBA and BCCC (with outer code the $(5,7)_8$ and inner accumulators) with that of their ergodic fading AWGN counterparts, namely repeat and accumulate (RA) and serial concatenated convolutional codes (SCCC), mapped over $N_B = 2$ fading blocks with 10 decoding iterations and 1024 information bits per frame. As we can observe, there is a significant difference in the error rate curves due to the blockwise concatenation which preserves δ_β of the outer code. Indeed, RA and SCCC codes are designed for the ergodic channel and, when transmitted over the block fading channel, they cannot exploit the available (limited) diversity.

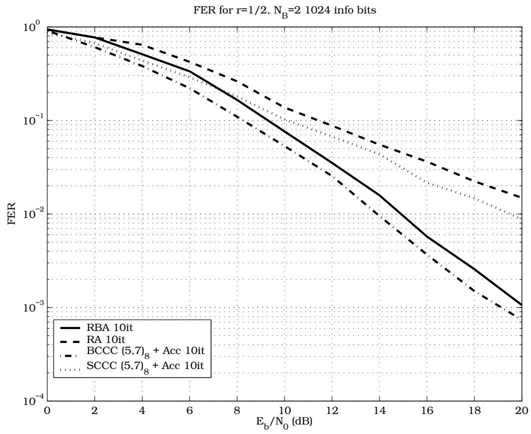

Figure 2: FER RBA, RA, BCCC and SCCC of $r = 1/2$ for $N_B = 2$ and 1024 information bits.

7.2. ML vs. Iterative Decoding

Figure 3 shows the simulation with 10 decoding iterations and ML bounds for RBA codes of $r = 1/2$ over $N_B = 2$ fading blocks for several block lengths. The statistical average in (3) and (4) is computed by Monte Carlo. For the sake of completeness we also plot the outage probability for both Gaussian and BPSK inputs. As we can observe, there is an excellent matching between iterative decoding and ML decoding, even for short block lengths, in contrast to the AWGN case. We can also see that using the tangential-sphere bound in a block-fading environment does not provide almost any significant gain over the limit before average union bound of [7]. Notice also the high computational cost of the average tangential-sphere bound: for each channel realization we have to optimize the cone radius, which makes it impractical for large block lengths. For all these aforementioned reasons, we conclude that the performance of BCCs can be very well predicted with simple techniques.

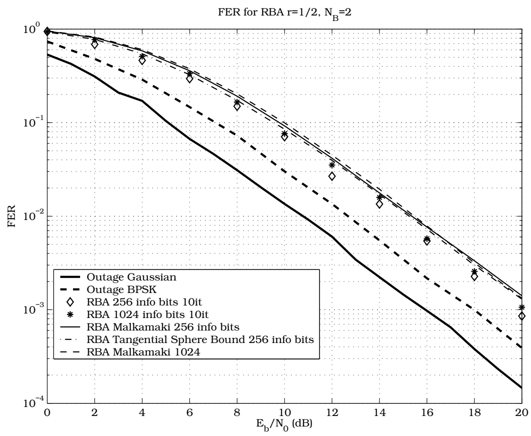

Figure 3: FER and ML bounds for RBA of $r = 1/2$ over $N_B = 2$ blocks.

7.3. Approaching Outage Probability

Figure 4 shows the FER as a function of the block length (in information bits) performance of $r = 1/4$ RBA codes and 64 states CC mapped over $N_B = 4$ blocks, and of $r = 1/2$ BCCC ($(5,7)_8$ and inner accumulators) and 64 states CC mapped over $N_B = 8$ blocks. As widely known and shown in the figure, the FER of convolutional codes increases with the block length while BCCs show flat performance. This effect can also be seen from the ML bounds reported in Figure 3, for which the bounds corresponding to 256 and 1024 information bits almost coincide. This very interesting effect constitutes one of the most important features of BCCs, since information theoretic limits, i.e., outage probability, can be approached in a *fair* way independent of the block length. This calls for optimizing the component encoders in order to make the resulting BCC as powerful as possible.

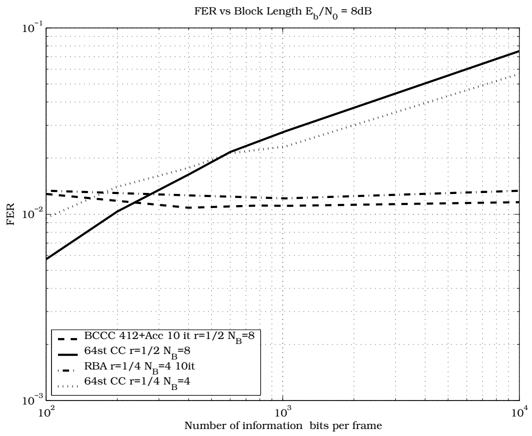

Figure 4: FER vs. input block length (information bits) at $E_b/N_0 = 8\text{dB}$ for BCCC, RBA and CC.

Figure 5 illustrates the FER performance of $r = 1/2$ BCCCs $(5,7)_8$ and $(25,35)_8$ both with inner accumulators, and best known 4 and 64 states CCs mapped over $N_B = 8$ fading blocks with block length of 1024 information bits. Notice that $(5,7)_8$ is not MDS, since $\delta_\beta = 4$ while the Singleton bound yields $\delta_\beta \leq 5$, and therefore the corresponding BCCC (and of course itself) will show some performance degradation at high SNR. Indeed, we can appreciate a steeper slope of the BCCC with $(25,35)_8$ and the 64 states CC (which are both MDS).

8. CONCLUSIONS AND APPLICATIONS

In this paper we have studied turbo-like codes over the block-fading channel, and we have shown how to construct such codes in order to achieve the diversity/rate tradeoff offered by the Singleton bound. We have analyzed their performance with ML bounding techniques and we have shown that iterative decoding performs very close to ML decoding even for short block lengths. The proposed codes are shown to approach outage probability in a fair manner, i.e., their FER performance is almost insensitive when increasing the block length. Blockwise concatenated codes appear very suited for data transmission with automatic repeat request (ARQ) protocols, and for variable-length packet transmission such as wireless IP, since they demonstrate very good FER performance irrespectively of the packet length.

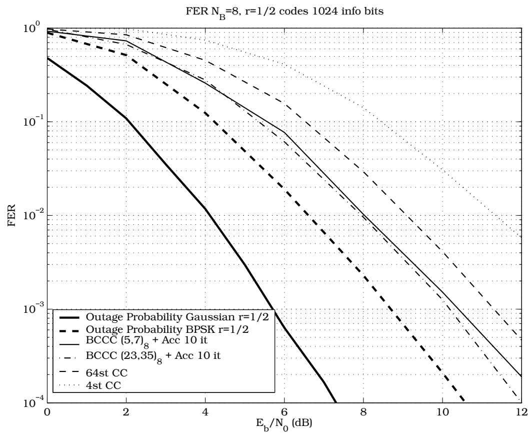

Figure 5: FER $r = 1/2$ BCCCs and CCs mapped over $N_B = 8$ fading blocks.

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